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2005 J. Phys. A: Math. Gen. 38 4391

(http://iopscience.iop.org/0305-4470/38/20/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 4391–4398 doi:10.1088/0305-4470/38/20/007

Invariant eigenoperators and energy gap for some Hamiltonians describing photonic nonlinear interaction

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Received 29 October 2004, in final form 4 April 2005
Published 3 May 2005
Online at stacks.iop.org/JPhysA/38/4391

Abstract
On the basis of the Heisenberg equation of motion and Schrödinger's correspondence $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \longleftrightarrow H$, we search for so-called invariant eigenoperators for some Hamiltonians describing photonic nonlinear interactions. In this way the energy-level gap of the Hamiltonians can be naturally obtained. The characteristic polynomial theory has been fully employed in our derivation.

PACS number: 05.45.−a

1. Introduction

Searching for the energy level gap of quantum mechanical dynamic systems has been a challenge to theoreticians since Bohr proposed atomic orbit theory [1]. Usually, solving the stationary Schrödinger equation [2] $H\psi_n = E_n\psi_n$ is the popular way to derive energy eigenvalues and eigenstates, where H does not include time explicitly. Instead of using the traditional method, in [3, 4] we have proposed the so-called 'invariant eigenoperator' method to deduce energy-level gap. This is based on the Heisenberg equation of motion [5]

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{O} = \frac{1}{\mathrm{i}}[\hat{O}, \hat{H}], \qquad \hbar = 1. \tag{1}$$

Although this equation, governing the time evolution of operators $\hat{O}$ for the time-independent Hamiltonian, plays the equivalent role of the Schrödinger equation, for a long time it has been seldom directly employed to derive energy quantization formulae. In paper [3] we have reported that the Heisenberg equation of motion can also be used to deduce the energy-level gap of certain systems in a transparent and concise way, provided that the 'eigenoperator' $\hat{O}_e$ of the n-power of the Schrödinger operator $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$ can be found. By invariant eigenoperator $\hat{O}_e$ we mean that it satisfies the following eigenvector-like equation:

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \hat{O}_e = \lambda \hat{O}_e. \tag{2}$$

One must not consider in general that the time derivation $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$ in equation (2) is the 'energy operator'. Nevertheless, using (1) and (2) we see

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \hat{O}_e = [\cdots[[\hat{O}_e, \hat{H}], \hat{H}]\cdots, \hat{H}] = \lambda \hat{O}_e, \tag{3}$$

where there are n-commutators on the right-hand side. In particular, when $n = 1$, according to what Schrödinger initiated, $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \longleftrightarrow H$, λ is then the energy eigenvalue. For this case the simplest example is $H = \omega a^\dagger a$, where $a^\dagger, a$ are Bose creation and annihilation operator, respectively, obeying $[a^\dagger, a] = -1$, from $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}a = [a, \hat{H}] = \omega a$, we can say that a is the 'eigenoperator' of $\omega a^\dagger a$, the energy-level gap is ω. In [3, 4] we mainly concentrated on the $n = 2$ case. Supposing $|c\rangle$ and $|b\rangle$ are two adjacent eigenstates of the Hamiltonian $\hat{H}$ with eigenenergy E_b and E_c, from (3) we see

$$\begin{aligned}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \langle c|\hat{O}_e|b\rangle &= \langle c|(\hat{O}_e\hat{H}^2 - 2\hat{H}\hat{O}_e\hat{H} + \hat{H}^2\hat{O}_e)|b\rangle \\ &= (E_b - E_c)^2\langle c|\hat{O}_e|b\rangle = \lambda\langle c|\hat{O}_e|b\rangle,\end{aligned} \tag{4}$$

because $\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \longleftrightarrow H^2$, we can judge that $\sqrt{\lambda} = E_b - E_c$ is one of the energy gaps. Equation (3) can then be viewed as a parallel equation with the energy eigenvector equation $H^2\psi = E^2\psi$. For example, when $H = \omega a^\dagger a + \mathrm{i}\lambda(a^2 - a^{\dagger 2})$, the Hamiltonian that describes a degenerate parametric amplifier, it is easy to evaluate

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 (a + a^\dagger) = (\omega^2 - 4\lambda^2)(a + a^\dagger),$$

so the eigenoperator is $(a + a^\dagger)$, and we immediately know that the energy gap is $\sqrt{\omega^2 - 4\lambda^2}$, without appealing to the usual diagonalization method for H. Generally speaking, due to $\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \longleftrightarrow H^n$, from (2) we can judge that $\sqrt[n]{\lambda}$ is one of the energy gap. Note that the value of λ in (2) depends on the power of the operator $\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n$, which is a linear transformation defined on a vector space composed of linear operators acting on the original Hilbert space. The powers of the linear map have the same eigenvectors (they commute), but the eigenvalues must be the powers of the original ones for the $\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)$. It turns out that the powers of $\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)$ can have degenerated eigenvalues, i.e. we can have new eigenvectors as well (linear combinations of the eigenvectors corresponding to the same eigenvalue). If H can be split into two parts,

$$H = H_0 + H' \tag{5}$$

and if

$$[\hat{O}_e, \hat{H}_0] = E_0\hat{O}_e, \qquad [\hat{O}_e, \hat{H}'] = E'\hat{O}_e, \tag{6}$$

then

$$[\hat{O}_e, \hat{H}] = (E_0 + E')\hat{O}_e. \tag{7}$$

That is, if the two Hamiltonians commute with each other, then they have the same invariant eigenoperator. Thus

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_e = [\hat{O}_e, H_0 + H'] \equiv (E_0 + E')\hat{O}_e. \tag{8}$$

Therefore, once the 'invariant eigenoperator' $\hat{O}_e$ is found, some information about the energy-level gap can be obtained. Before we examine the $n \geqslant 2$ case in more detail, in this paper we present some complicated examples in the $n = 1$ case to elucidate the validity of our approach. These examples are from nonlinear optics.

2. Invariant eigenoperator for the 3-wave mixing model

The first example is about a 3-wave mixing model in quantum optics or in nonlinear optics theory [6]; its Hamiltonian is

$$H = \sum_{i=1}^{3} \omega_i a_i^{\dagger} a_i + \kappa \left(a_1^{\dagger} a_2 a_3 + \text{H.C.} \right), \tag{9}$$

where H.C. means the Hermitian conjugate of $a_1^{\dagger} a_2 a_3$, $\left[a_i, a_j^{\dagger}\right] = \delta_{ij}$. For energy conservation, it is demanded that $\omega_1 = \omega_2 + \omega_3$; then it follows that

$$H = H_0 + H', \qquad H_0 \equiv \sum_{i=1}^{3} \omega_i a_i^{\dagger} a_i, \qquad H' \equiv \kappa \left(a_1^{\dagger} a_2 a_3 + \text{H.C.} \right), \qquad [H_0, H'] = 0. \tag{10}$$

We hope to find the common eigenoperators of H_0 and H'. In $n = 1$ case of (3), the general form of the eigenoperator of H_0 is $a_1^{\dagger m} a_2^{\dagger n} a_3^{\dagger k}$, since

$$\left[a_1^{\dagger m} a_2^{\dagger n} a_3^{\dagger k}, H_0 \right] = \left[a_1^{\dagger m} a_2^{\dagger n} a_3^{\dagger k}, \sum_{i=1}^{3} \omega_i a_i^{\dagger} a_i \right] = -(m\omega_1 + n\omega_2 + k\omega_3) a_1^{\dagger m} a_2^{\dagger n} a_3^{\dagger k}. \tag{11}$$

Thus the common eigenoperator of H_0 and H_1 is a polynomial function made of $a_1^{\dagger}$, $a_2^{\dagger}$ and $a_3^{\dagger}$; since the energy arising from $a_1^{\dagger}$ can be denoted by that of the other two modes due to $\omega_1 = \omega_2 + \omega_3$, we only need two independent parameters to denote such an eigenoperator. Moreover, from the interacting form of $a_1^{\dagger} a_2 a_3$ we see that the creation of the first mode photon always accompanies the annihilation of the other two modes of photons, due to $\left(a_i^{\dagger}\right)^{-1}$ $(i = 2, 3)$ meaning annihilating operation, so without loss of generality we assume that the invariant eigenoperator of $H_0 + H'$ in 3-mode Fock space is an operator polynomial in the form,

$$\hat{O}_e = \sum_{l=0}^{q} F(l) a_1^{\dagger l} a_2^{\dagger \mu - l} a_3^{\dagger \nu - l} |000\rangle\langle 000|, \tag{12}$$

where $F(l)$, as one can see later, is determined by $[\hat{O}_e, H'] \equiv E' \hat{O}_e$ (as the eigenoperator is not in general a polynomial function of the operators from equation (12), but only those that have the same eigenvalues), and is related to μ and ν. μ, ν are arbitrarily chosen positive integer parameters; q is not larger than the smallest of μ and ν. Rewriting

$$\hat{O}_e \equiv \Sigma\Lambda, \qquad \Sigma \equiv \sum_{l=0}^{q} F(l) a_1^{\dagger l} a_2^{\dagger \mu - l} a_3^{\dagger \nu - l}, \qquad \Lambda \equiv |000\rangle\langle 000|, \tag{13}$$

from $[H_0, \Lambda] = 0$, we see that

$$[\hat{O}_e, H_0] = \left[\Sigma, \sum_{i=1}^{3} \omega_i a_i^{\dagger} a_i \right] \Lambda = E_0 \hat{O}_e, \qquad E_0 = -(\mu\omega_2 + \nu\omega_3). \tag{14}$$

Note that the vacuum projector $|000\rangle\langle 000|$ in (12) is also an operator polynomial,

$$|000\rangle\langle 000| =: \exp\left[-\sum_{i=1}^{3} a_i^{\dagger} a_i \right] := \prod_{i=1}^{3} \sum_{n=0}^{\infty} \frac{(-)^n a_i^{\dagger n} a_i^n}{n!}, \tag{15}$$

where : : denotes normal ordering. According to (8) it remains to examine

$$[\hat{O}_e, H'] \equiv E' \hat{O}_e. \tag{16}$$

Due to

$$[\Lambda, H'] = \left[|000\rangle\langle 000|, \kappa\left(a_1^\dagger a_2 a_3 + a_1 a_2^\dagger a_3^\dagger\right)\right] = 0, \tag{17}$$

and $a_i\Lambda = 0$ as well as $\left[a_i, f\left(a_i^\dagger\right)\right] = \frac{\partial f}{\partial a_i^\dagger}$, we can derive

$$\begin{aligned}[\hat{O}_e, H'] &= [\Sigma, H']\Lambda \\ &= -\kappa a_1^\dagger[a_2a_3, \Sigma]\Lambda - \kappa a_2^\dagger a_3^\dagger[a_1, \Sigma]\Lambda \\ &= -\kappa a_1^\dagger\left(\frac{\partial^2\Sigma}{\partial a_2^\dagger \partial a_3^\dagger}\right)\Lambda - \kappa a_2^\dagger a_3^\dagger \frac{\partial \Sigma}{\partial a_1^\dagger}\Lambda.\end{aligned} \tag{18}$$

Substituting (18) into (16) we have

$$a_1^\dagger \frac{\partial^2\Sigma}{\partial a_2^\dagger \partial a_3^\dagger}\Lambda + a_2^\dagger a_3^\dagger \frac{\partial\Sigma}{\partial a_1^\dagger}\Lambda = -\frac{E'}{\kappa}\Sigma\Lambda. \tag{19}$$

Using the expression of Σ in (16) and performing the differentiation in (19) we see that the coefficient $p_l \equiv l!F(l)$ should satisfy the difference equation,

$$p_{l+1} = -\frac{E'}{\kappa}p_l - s_l p_{l-1}, \tag{20}$$

where $F_{(-1)} = F_{(q+1)} = 0$,

$$s_l = l(\mu - l + 1)(\nu - l + 1), \tag{21}$$

i.e. p_l is an polynomial sequence obeying the three-term recurrence relation. At this point, we introduce the matrix

$$\begin{pmatrix} c_0 & b_1 & & \\ 1 & c_1 & b_2 & \\ & 1 & c_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \equiv A \tag{22}$$

where the rows and columns of the matrix are indexed by nonnegative integers. Let A_{l+1} denote the square matrix obtained from A but taking the first $l+1$ rows and columns. Observe that when $\det(xI - A_{l+1})$ is expanded about the last row, we get

$$\det(xI - A_{l+1}) = (xI - c_l)\det(XI - A_1) - b_l\det(XI - A_{(I-1)}), \tag{23}$$

where I is the identity matrix.

Thus $p_{l+1}(x)$ can be considered the characteristic polynomial of A_{l+1}. By comparison (20) with (23) we see that the coefficient p_l in (20) can be written as

$$p_{l+1} = \det\left(-\frac{E'}{\kappa}I - A_{l+1}\right), \tag{24}$$

where

$$A_{l+1} = \begin{pmatrix} 0 & s_1 & & \\ 1 & 0 & s_2 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}_{(l+1)\times(l+1)}. \tag{25}$$

Thus for arbitrarily selected positive integers μ and ν, we use (21) and (25) to construct the matrix A_{q+1}; due to $\det\left(-\frac{E'}{\kappa}I - A_{q+1}\right) = 0$, we can derive $q+1$ values of E', then for each value of E' we can obtain $p_0, p_1, \ldots, p_q$ by $p_l = \det\left(-\frac{E'}{\kappa}I - A_l\right)$, i.e. we can determine the characteristic polynomial p_l and $F(l)$. (Note that $p_0 = 1$ to ensure (20) holding for $I = 0$.) Finally, using (12) the eigenoperator can be found, and simultaneously the energy E' is known.

3. Invariant eigenoperator for the 4-wave mixing model

To see our method more clearly, we discuss the 4-wave mixing Hamiltonian which has the form [6, 7]

$$\mathcal{H} = \sum_{i=1}^{4} \omega_i a_i^\dagger a_i + g\big(a_1^\dagger a_2 a_3 a_4 + \text{H.C.}\big). \tag{26}$$

For energy conservation, we have demanded that $\omega_1 = \omega_2 + \omega_3 + \omega_4$; then it follows that

$$\mathcal{H}_0 \equiv \sum_{i=1}^{4} \omega_i a_i^\dagger a_i, \qquad \mathcal{H}' \equiv g\big(a_1^\dagger a_4 a_2 a_3 + \text{H.C.}\big), \qquad [\mathcal{H}_0, \mathcal{H}'] = 0. \tag{27}$$

We try to find the common eigenoperator of $\mathcal{H}_0$ and $\mathcal{H}'$. In similar to (12) we assume that the invariant eigenoperator $\hat{O}_e'$ of $\mathcal{H}$ takes the form,

$$\hat{O}_e' = \Sigma\Lambda, \tag{28}$$

where

$$\Sigma \equiv \sum_{l=0}^{q'} F'(l) a_1^{\dagger l} a_2^{\dagger \mu - l} a_3^{\dagger \nu - l} a_4^{\dagger \tau - l}, \qquad \Lambda \equiv |0000\rangle\langle 0000|, \tag{29}$$

where $F'(l)$ is determined by $[\hat{O}_e', \mathcal{H}'] = \lambda' \hat{O}_e'$ and is related to μ, ν, τ, q' is not more than the smallest of μ, ν, τ. The corresponding Heisenberg equation for $\hat{O}_e'$ is

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_e' = [\hat{O}_e', \mathcal{H}_0 + \mathcal{H}'] \equiv (\lambda_0 + \lambda')\hat{O}_e' \tag{30}$$

from (29) and (27) we have

$$[\hat{O}_e', \mathcal{H}_0] = \lambda_0 \hat{O}_e', \qquad \lambda_0 = -(\mu\omega_2 + \nu\omega_3 + \tau\omega_4). \tag{31}$$

Further, from $[\mathcal{H}', \Lambda] = 0$ and $a_i\Lambda = 0$ we can derive

$$\begin{aligned}
[\mathcal{H}', \hat{O}_e'] &= [\mathcal{H}', \Sigma]\Lambda \\
&= g a_1^\dagger \{a_2[a_3, \Sigma]a_4 + a_2 a_3[a_4, \Sigma] + [a_2, \Sigma]a_3 a_4\}\Lambda + g a_2^\dagger a_3^\dagger a_4^\dagger [a_1, \Sigma]\Lambda \\
&= g a_1^\dagger a_2 a_3 [a_4, \Sigma]\Lambda + g a_2^\dagger a_3^\dagger a_4^\dagger [a_1, \Sigma]\Lambda \\
&= g a_1^\dagger \left(\frac{\partial^3 \Sigma}{\partial a_2^\dagger \partial a_3^\dagger \partial a_4^\dagger}\right)\Lambda + g\left(a_2^\dagger a_3^\dagger a_4^\dagger \frac{\partial \Sigma}{\partial a_1^\dagger}\right)\Lambda.
\end{aligned} \tag{32}$$

Substituting (32) into $[\mathcal{H}', \hat{O}_e'] = -\lambda' \hat{O}_e'$ we obtain

$$a_1^\dagger \frac{\partial^3 \Sigma}{\partial a_2^\dagger \partial a_3^\dagger \partial a_4^\dagger}\Lambda + a_2^\dagger a_3^\dagger a_4^\dagger \frac{\partial \Sigma}{\partial a_1^\dagger}\Lambda = -\frac{\lambda'}{g}\Sigma\Lambda. \tag{33}$$

Using the expression of Σ in (29) and performing the differentiation in (33) we see that the coefficient $p_l' \equiv F'(l) l!$ should satisfy the difference equation

$$p_{l+1}' = -\frac{\lambda'}{g} p_l' - s_l' p_{l-1}', \qquad s_l' = (\mu - l + 1)(\nu - l + 1)(\tau - l + 1) l. \tag{34}$$

p_l' is the characteristic polynomial of A_l',

$$p_l' = \det\left(-\frac{\lambda'}{g} I - A_l'\right), \tag{35}$$

where

$$A'_l = \begin{pmatrix} 0 & s'_1 & & \\ 1 & 0 & s'_2 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}_{l\times l}. \tag{36}$$

For arbitrarily selected μ, ν, τ and using (34) we construct the matrix $A'_{q'+1}$; solving $\det\left(-\frac{\lambda'}{g}I - A'_{q'+1}\right) = 0$, we can derive the $q'+1$ values of λ'. Then for each value of λ' we can obtain $p'_0, p'_1, \ldots, p'_{q'}$ by $p'_l = \det\left(-\frac{\lambda'}{g}I - A'_l\right)$, i.e. we can determine the characteristic polynomial p'_l and $F'(l)$. In this way the eigenoperator and energy gap can be found. Now we present some examples.

Assuming that μ is the smallest among μ, ν, τ, then from (34) we see

(1) When $\mu = 1, q' = 1$,

$$A'_2 = \begin{pmatrix} 0 & s'_1 \\ 1 & 0 \end{pmatrix}, \qquad \text{where} \quad s'_1 = \nu\tau,$$
$$\det\left(-\frac{\lambda'}{g}I - A'_2\right) = \det\begin{pmatrix} -\frac{\lambda'}{g} & -s'_1 \\ -1 & -\frac{\lambda'}{g} \end{pmatrix} = 0, \tag{37}$$

$\Rightarrow -\frac{\lambda'}{g} = \pm\sqrt{\nu\tau}$.

For $-\frac{\lambda'}{g} = \sqrt{\nu\tau}$, $p'_0 = 1$, $p'_1 = \sqrt{\nu\tau}$, the invariant eigenoperator is

$$\hat{O}'_e = \left(a_2^\dagger a_3^{\dagger\nu} a_4^{\dagger\tau} + \sqrt{\nu\tau} a_1^\dagger a_3^{\dagger\nu-1} a_4^{\dagger\tau-1}\right)|0000\rangle\langle 0000|, \tag{38}$$

and the energy gap is

$$\Delta E = \lambda_0 + \lambda' = -(\mu\omega_2 + \nu\omega_3 + \tau\omega_4) - g\sqrt{\nu\tau}. \tag{39}$$

For $-\frac{\lambda'}{g} = -\sqrt{\nu\tau}$,

$$\hat{O}'_e = \left(a_2^\dagger a_3^{\dagger\nu} a_4^{\dagger\tau} - \sqrt{\nu\tau} a_1^\dagger a_3^{\dagger\nu-1} a_4^{\dagger\tau-1}\right)|0000\rangle\langle 0000|, \tag{40}$$

the energy gap is

$$\Delta E = -(\mu\omega_2 + \nu\omega_3 + \tau\omega_4) + g\sqrt{\nu\tau}. \tag{41}$$

(2) When $\mu = 2, q' = 2$,

$$A'_3 = \begin{pmatrix} 0 & s'_1 & 0 \\ 1 & 0 & s'_2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \text{where} \quad s'_1 = 2\nu\tau, \quad s'_2 = 2(\nu-1)(\tau-1),$$
$$\det(\lambda' I - A'_3) = \det\begin{pmatrix} \lambda' & -s'_1 & 0 \\ -1 & \lambda' & -s'_2 \\ 0 & -1 & \lambda' \end{pmatrix} = 0, \tag{42}$$

$\Rightarrow \left(-\frac{\lambda'}{g}\right)^3 - \left(-\frac{\lambda'}{g}\right)s'_1 - \left(-\frac{\lambda'}{g}\right)s'_2 = 0, \Rightarrow -\frac{\lambda'}{g} = 0, \pm(s'_1 + s'_2)^{1/2}$.

For $-\frac{\lambda'}{g} = 0$, $p'_0 = 1$, $p'_1 = 0$, $p'_2 = 2\nu\tau$, so the invariant eigenoperator is

$$\hat{O}'_e = \left(a_2^{\dagger 2} a_3^{\dagger\nu} a_4^{\dagger\tau} - \nu\tau a_1^{\dagger 2} a_3^{\dagger\nu-2} a_4^{\dagger\tau-2}\right)|0000\rangle\langle 0000|, \tag{43}$$

the energy gap is

$$\Delta E = -(\mu\omega_2 + \nu\omega_3 + \tau\omega_4). \tag{44}$$

For $-\frac{\lambda'}{g} = (s_1' + s_2')^{1/2}$, $p_0' = 1$, $p_1' = (s_1' + s_2')^{1/2}$, $p_2' = s_2'$,

$$\hat{O}_e' = \left(a_2^{\dagger 2} a_3^{\dagger \nu} a_4^{\dagger \tau} + (s_1' + s_2')^{1/2} a_1^\dagger a_2^\dagger a_3^{\dagger \nu - 1} a_4^{\dagger \tau - 1} + \frac{s_2'}{2} a_1^{\dagger 2} a_3^{\dagger \nu - 2} a_4^{\dagger \tau - 2} \right) |0000\rangle\langle 0000|, \tag{45}$$

the energy gap is

$$\begin{aligned} \Delta E &= -(\mu\omega_2 + \nu\omega_3 + \tau\omega_4) - (s_1' + s_2')^{1/2} \\ &= -(\mu\omega_2 + \nu\omega_3 + \tau\omega_4) - \sqrt{4\nu\tau - 2\tau - 2\nu + 2}. \end{aligned} \tag{46}$$

For $-\frac{\lambda'}{g} = -(s_1' + s_2')^{1/2}$, $p_0' = 1$, $p_1' = -(s_1' + s_2')^{1/2}$, $p_2' = s_2'$, the operator is

$$\hat{O}_e' = \left(a_2^{\dagger 2} a_3^{\dagger \nu} a_4^{\dagger \tau} - (s_1' + s_2')^{1/2} a_1^\dagger a_2^\dagger a_3^{\dagger \nu - 1} a_4^{\dagger \tau - 1} + \frac{s_2'}{2} a_1^{\dagger 2} a_3^{\dagger \nu - 2} a_4^{\dagger \tau - 2} \right) |0000\rangle\langle 0000|, \tag{47}$$

the energy gap is

$$\Delta E = -(\mu\omega_2 + \nu\omega_3 + \tau\omega_4) + \sqrt{4\nu\tau - 2\tau - 2\nu + 2}. \tag{48}$$

Note that the factor $-(\mu\omega_2 + \nu\omega_3 + \tau\omega_4)$ appears in (41), (44), (46) and (48), because it is the contribution from $\mathcal{H}_0$. The rest terms in (41), (46) and (48) respectively depend on the photon number ν and τ which originate from Σ in (29) when μ is fixed, so for different eigenoperators $\Sigma\Lambda$ we obtain different energy gaps. This fact tells us that for optical nonlinear interaction the information on the energy gap is abundant.

4. Discussion

We can extend the above cases to the n-wave mixing Hamiltonian,

$$H = \sum_{i=1}^{n} \omega_i a_i^\dagger a_i + \left(\kappa a_1^\dagger a_2 a_3 \cdots a_n + \text{H.C.} \right), \qquad \omega_1 = \sum_{i=2}^{n} \omega_i. \tag{49}$$

The eigenoperator of H is

$$\hat{O}_e = \sum_{l=0}^{Q} \frac{p_l}{l!} a_1^{\dagger l} a_2^{\dagger \mu_1 - l} a_3^{\dagger \mu_2 - l} \cdots a_n^{\dagger \mu_{n-1} - l} |00\cdots 0\rangle\langle 00\cdots 0|, \tag{50}$$

where $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are arbitrarily chosen positive integers, Q is the smallest of them. We can prove that p_l satisfies the relation

$$\begin{aligned} p_l &= \lambda' p_{l-1} - s_{l-1} p_{l-2} = \det(\lambda' I - A_l), \\ s_l &= (\mu_1 - l + 1)(\mu_2 - l + 1) \cdots (\mu_{n-1} - l + 1) l. \end{aligned} \tag{51}$$

By analogy with the derivations in the previous section it is possible to derive the H' invariant eigenoperator and the energy-level gap.

In summary, in this work we have elucidated the 'invariant eigenoperator' method for deriving the energy gap of some Hamiltonians describing photonic nonlinear interaction, which is a direct application of the well-known Heisenberg equation. In our approach the characteristic polynomial theory has been fully employed. These models have also been discussed in [8], in which the authors used the stationary Schrödinger equation $H\psi_n = E_n\psi_n$ to search for eigenstates of the Hamiltonian. As one can see from [3, 4], in many cases the 'invariant eigenoperator' method, stemming from the Heisenberg approach, is more direct in obtaining the energy-level gap formula than the Schrödinger approach.

Acknowledgments

This work is supported by National Natural Science Foundation of China under grant 10475056 and Specialized Research Fund for the Doctorial Program of Higher Education (SRFDP) as well as the Foundation of the President of Chinese Academy of Science.

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